Determinants of Matrices from Pascal's Triangle

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Abstract

When devising problems for a linear algebra course it is desirable to have an extensive source of integral matrices of determinant one. In this short paper we give a straightforward method of generating some such matrices from binomial coefficients.

1 Introduction

We present a simple result on the determinant of certain matrices devised from Pascal's triangle. The inspiration behind this result was the quest for a ready supply of suitable matrices for linear algebra problem sets. The goal being integral matrices which can be row reduced without introducing fractions so that one can test knowledge of row and column operations whilst minimising the risk of computational errors. In practice, this means finding integral matrices with determinant ± 1.

Experimentation led to the conjecture that the following procedure worked: write Pascal's triangle in left-justified form and choose a square block which abuts the left-hand edge.

Using the theory of numerical polynomials we are able to prove an expanded version of this initial conjecture. The procedure now works as follows:

1. Write Pascal's triangle in left-justified form. We label the columns starting with zero so that the kth column corresponds to $\binom{n}{k}$:

$$\begin{matrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 2 & 1 & \cdots \\ 1 & 3 & 3 & \cdots \\ 1 & 4 & 6 & \cdots \end{matrix}$$

2. Extend the table in the vertical direction by applying the definition

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!}$$

also to negative n. We have the useful symmetry formula:

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}.$$

So a section of the table is:

$$\begin{matrix} 1 & -2 & 3 & \cdots \\ 1 & -1 & 1 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 2 & 1 & \cdots \\ 1 & 3 & 3 & \cdots \\ 1 & 4 & 6 & \cdots \end{matrix}$$

3. Choose a size of matrix, say k.

4. Shift each of the columns $1, \ldots, k-1$ vertically by some arbitrary amount (technically one can do this for the zeroth column as well but of course it makes no difference).

$$\begin{matrix} 1 & 3 & 0 & \cdots \\ 1 & 4 & 0 & \cdots \\ 1 & 5 & 1 & \cdots \\ 1 & 6 & 3 & \cdots \\ 1 & 7 & 6 & \cdots \end{matrix}$$

5. From the resulting matrix, choose a $k \times k$ block which abuts the left-hand edge:

$$\begin{bmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 1 & 7 & 6 \end{bmatrix}$$

6. This matrix will have determinant 1:

$$\begin{aligned} 1 \cdot 6 \cdot 6 + 5 \cdot 3 \cdot 1 + 1 \cdot 1 \cdot 7 - 1 \cdot 3 \cdot 7 - 5 \cdot 1 \cdot 6 - 1 \cdot 6 \cdot 1 \\ &= 36 + 15 + 7 - 21 - 30 - 6 \\ &= 1. \end{aligned}$$

We actually go a little further than this and examine what happens if after shifting the columns one selects elements from each column periodically, with the period varying with the column. For example, one might take every 3rd element from the first column, every 5th from the second, and so on. The resulting determinant depends only on these selection rules and the order in which they are applied. Specifically, if in the jth column one takes every q_jth term the resulting matrix has determinant $\prod q_j{}^j$.

In Section 2 we will introduce the theory of Numerical Polynomials. These are well-known, but we shall prove the results we need for completeness' sake. We will end this section with a general theorem on matrices which we will use in Section 3 to prove the result described above and some generalisations. Finally, in Section 4 we will look at a few ideas suggested by the investigation.

2 Numerical Polynomials

The key theorem is more general than the above discussion implies, although it is not complicated. We have concentrated in the introduction on the application to binomial coefficients due to the original motivation and due to their simplicity. The proof depends on the structure of the ring $\mathrm{Int}(\mathbb{Z})$ of rational polynomials which map $\mathbb{Z}$ into itself. This ring has been extensively studied and much is known about it. It has many applications outside number theory in the field of algebraic topology which is where I encountered it, although that has little to do with this result.

The work in this paper depends on the following well-known result which we will prove for completeness:

Proposition 2.1 *For $k \geq 0$, define the kth* Newton *polynomial $b_k(w) \in \mathbb{Q}[w]$ by:*

$$b_k(w) := \frac{w(w-1)\cdots(w-k+1)}{k!}$$

then $b_k(w) \in \mathrm{Int}(\mathbb{Z})$ and the sequence $(b_k(w))$ is a basis for $\mathrm{Int}(\mathbb{Z})$ as a $\mathbb{Z}$–module.

Proof. That $b_k(w)$ lies in $\mathrm{Int}(\mathbb{Z})$ comes from the fact that its value at $n \in \mathbb{N}$ is the integer $\binom{n}{k}$, whilst at negative n, its value is $(-1)^k\binom{k-n-1}{k}$, also an integer.

As $b_k(w)$ has degree k, the sequence $(b_k(w))$ is a $\mathbb{Q}$–basis for $\mathbb{Q}[w]$. Therefore there cannot be any relations between the $b_k(w)$ in $\mathbb{Q}[w]$ and hence neither can there be any relations between them in $\mathrm{Int}(\mathbb{Z})$. Hence the $\mathbb{Z}$–submodule of $\mathrm{Int}(\mathbb{Z})$ spanned by the sequence $(b_k(w))$ is free and has $(b_k(w))$ as a $\mathbb{Z}$–basis. It remains to show that this is the whole of $\mathrm{Int}(\mathbb{Z})$.

We need to show that a non-zero element in $\mathrm{Int}(\mathbb{Z})$ can be written as a $\mathbb{Z}$–linear combination of the $b_k(w)$. As the $b_k(w)$ are a $\mathbb{Q}$–basis for $\mathbb{Q}[w]$ we already know that any such element can be written as a $\mathbb{Q}$–linear combination of the $b_k(w)$. We just need to show that the coefficients have to be integers. Define the *length* of a non-zero polynomial $f(w) \in \mathrm{Int}(\mathbb{Z})$ to be the smallest integer l such that $f(w)$ is a $\mathbb{Q}$–linear combination of $\{b_k(w), \dots, b_{k+l}(w)\}$ for some k. Thus:

$$f(w) = c_k b_k(w) + \dots + c_{k+l} b_{k+l}(w)$$

for $c_k \in \mathbb{Q}$ with c_k and c_{k+l} non-zero. We shall proceed by induction on this value.

For the initial step, let $f(w) \in \mathrm{Int}(\mathbb{Z})$ have length zero. Then there is some k and $c_k \in \mathbb{Q}$ such that $f(w) = c_k b_k(w)$. Now $b_k(k) = \binom{k}{k} = 1$ so evaluating this at k yields the identity $c_k = f(k)$ which, as $f(w) \in \mathrm{Int}(\mathbb{Z})$, lies in $\mathbb{Z}$. Hence $f(w)$ is a $\mathbb{Z}$–linear combination of the $b_k(w)$.

Now suppose that whenever $g(w) \in \mathrm{Int}(\mathbb{Z})$ has length at most l then $g(w)$ lies in the $\mathbb{Z}$–linear span of the $b_k(w)$. Let $f(w) \in \mathrm{Int}(\mathbb{Z})$ have length $l+1$ and write:

$$f(w) = c_k b_k(w) + \dots + c_{k+l+1} b_{k+l+1}(w),$$

where $k \geq 0$ is chosen appropriately. Evaluate this at k. On the left-hand side we have $f(k)$ which is an integer. On the right-hand side the first term evaluates to c_k as $b_k(k) = 1$. The rest of the terms evaluate to zero as $b_{k+i}(k) = \binom{k}{k+i} = 0$ for $i > 0$. Hence $c_k = f(k) \in \mathbb{Z}$. Let $g(w) = f(w) - c_k b_k(w)$. This lies in $\mathrm{Int}(\mathbb{Z})$ and has length at most l whence is in the $\mathbb{Z}$–linear span of the $b_k(w)$. Consequently, so is $f(w)$. □

One way to think of this basis is as follows. There is an obvious map $\mathrm{Int}(\mathbb{Z}) \to \mathrm{Map}(\mathbb{Z}, \mathbb{Z})$, the space of self-maps of $\mathbb{Z}$. The dual of this space[1] has an obvious basis given by the evaluation maps

$$e_k \colon \mathrm{Map}(\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}, \quad e_k(f) = f(k).$$

The dual of this basis is therefore the delta maps $\delta_k \colon \mathbb{Z} \to \mathbb{Z}$,

$$\delta_k(j) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

These are obviously not polynomials, and the $(b_k(w))$ are the closest we can get as they satisfy:

$$b_k(j) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k > j \end{cases}$$

A particularly useful property of this basis is that it is filtered by degree. This means that when writing $f \in \mathrm{Int}(\mathbb{Z})$ as a $\mathbb{Z}$–linear combination of the $(b_k(w))$ we will only need those $b_k(w)$ with $k \leq \deg f$.

The fact that $b_k(w)$ has degree k means that $(b_k(w))$ is also a $\mathbb{Q}$–basis for $\mathbb{Q}[w]$ and an $\mathbb{R}$–basis for $\mathbb{R}[w]$. Although our primary interest is in $\mathrm{Int}(\mathbb{Z})$, our results will hold for these spaces as well, with the coefficient ring chosen as appropriate.

Thus if we have a sequence of polynomials, $(f_k(w))$, in one of $\mathrm{Int}(\mathbb{Z})$, $\mathbb{Q}[w]$, or $\mathbb{R}[w]$, such that $f_k(w)$ is of degree k and we gather together all the coefficients into a matrix equation:

$$\begin{bmatrix} f_0(w) \\ f_1(w) \\ f_2(w) \\ \vdots \end{bmatrix} = \begin{bmatrix} c_{ij} \end{bmatrix} \begin{bmatrix} b_0(w) \\ b_1(w) \\ b_2(w) \\ \vdots \end{bmatrix}$$

then the coefficient matrix $\begin{bmatrix} c_{ij} \end{bmatrix}$ is lower triangular (with coefficients in the appropriate coefficient ring). We can therefore truncate this at any $k \geq 0$ to produce:

$$\begin{bmatrix} f_0(w) \\ f_1(w) \\ \vdots \\ f_k(w) \end{bmatrix} = \begin{bmatrix} c_{ij} \end{bmatrix} \begin{bmatrix} b_0(w) \\ b_1(w) \\ \vdots \\ b_k(w) \end{bmatrix}$$

[1] Technically, we should be flinging the word "topology" around at this point.

with $\left[c_{ij}\right]$ still lower triangular.

This holds as an equation for polynomials so it also holds if we evaluate at any point. Thus if we choose a finite sequence of numbers $(q_0, \dots, q_k)$ we see that:

$$\begin{bmatrix} f_0(q_0) & f_0(q_1) & \cdots & f_0(q_k) \\ f_1(q_0) & f_1(q_1) & \cdots & f_1(q_k) \\ \dots & \dots & \dots & \dots \\ f_k(q_0) & f_k(q_1) & \cdots & f_k(q_k) \end{bmatrix} = \left[c_{ij}\right] \begin{bmatrix} b_0(q_0) & b_0(q_1) & \cdots & b_0(q_k) \\ b_1(q_0) & b_1(q_1) & \cdots & b_1(q_k) \\ \dots & \dots & \dots & \dots \\ b_k(q_0) & b_k(q_1) & \cdots & b_k(q_k) \end{bmatrix}.$$

As the matrix $\left(c_{ij}\right)$ is lower triangular it is simple to read off the determinant of the matrix $\left(f_i(q_j)\right)$ in terms of the determinant of the matrix $\left(b_i(q_j)\right)$.

Theorem 2.2 *Let $(f_0(w), \dots, f_k(w))$ be a finite sequence of polynomials in one of $\mathrm{Int}(\mathbb{Z})$, $\mathbb{Q}[w]$, or $\mathbb{R}[w]$ such that $f_i(w)$ has degree i. Let $(q_0, \dots, q_k)$ be a sequence of numbers. Then the determinant of the matrix $(f_i(q_j))$ is the product of $\prod_{j=0}^{k} f_j^{(j)}(0)$ and the determinant of the matrix $(b_i(q_j))$, where $b_i(w)$ is the ith Newtonian polynomial.*

Proof. All we need to show is that the diagonal of the coefficient matrix is the sequence $(f_j^{(j)}(0))$. The number c_{jj} is the coefficient of $b_j(w)$ in the expression of $f_j(w)$ as a linear combination of $\left\{b_0(w), \dots, b_j(w)\right\}$. As $b_i(w)$ has degree i we can kill off all the lower terms by taking the jth derivative on each side to leave $f_j^{(j)}(w) = c_{jj} b_j^{(j)}(w)$. A simple calculation reveals that $b_j^{(j)}(w)$ is the constant polynomial 1 and hence $f_j^{(j)}(w)$ is the constant polynomial c_{jj}. Equivalently, $c_{jj} = f_j^{(j)}(q)$ for any number q; in particular, $c_{jj} = f_j^{(j)}(0)$. □

Note that although we express the determinant in terms of the jth derivative of $f_j(w)$ this is just a fancy way of talking about $j!$ times by the coefficient of w^j in $f_j(w)$. It is often simplest to calculate this directly.

3 Shifting Pascal's Triangle

We now put in to Theorem 2.2 various sequences of polynomials to determine the determinants of the corresponding matrices. The easiest sequence of integers at which to evaluate our polynomials is $(0, 1, \dots, k)$. This is because the matrix $(b_i(j))_{0 \le i,j \le k}$ is upper triangular with leading diagonal all 1s and hence has determinant 1.

Corollary 3.1 *Let $k \in \mathbb{N}$ and let $q_1, \dots, q_k, n_0, \dots, n_k \in \mathbb{Z}$. Let B be the $(k+1) \times (k+1)$ matrix:*

$$B = \begin{bmatrix} 1 & n_1 & \binom{n_2}{2} & \dots & \binom{n_k}{k} \\ 1 & n_1 + q_1 & \binom{n_2+q_2}{2} & \dots & \binom{n_k+q_k}{k} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n_1 + q_1 k & \binom{n_2+q_2 k}{2} & \dots & \binom{n_k+q_k k}{k} \end{bmatrix},$$

then $\det B = \prod_{i=1}^{k} q_i{}^i$. *In particular, if* $q_i = 1$ *for all* i *then* $\det B = 1$ *and if* $q_i = q$ *is the same for all* i *then* $\det B = q^{\frac{1}{2}k(k+1)}$.

Proof. First we observe that the matrix $(b_i(j))$ is upper triangular with leading diagonal 1s. Hence it has determinant 1.

Let $f_0(w) = 1$ and for $1 \le i \le k$, let $f_i(w) = b_i(q_i w + n_i)$. As $\mathrm{Int}(\mathbb{Z})$ is closed under composition, $f_i(w) \in \mathrm{Int}(\mathbb{Z})$. Clearly $f_i(w)$ is of degree i and, by the chain rule, $f_i^{(i)}(0) = q_i{}^i$. Hence the matrix $(f_i(j))_{0\le i,j\le k}$ has determinant $\prod q_i{}^i$. By construction, $f_i(j) = \binom{n_i+q_i j}{i}$ and so the matrix B is the transpose of $(f_i(j))_{0\le i,j\le k}$. □

The column of 1s in this matrix is mildly irritating. We can generalise Theorem 2.2 to deal with this.

Definition 3.2 *Let* $f \in \mathrm{Int}(\mathbb{Z})$. *Define the* b*–*support *of* f, $b\text{–supp}(f)$, *to be the smallest set* $b\text{–supp}(f) \subset \mathbb{N}$ *such that* f *is a* $\mathbb{Z}$*–linear combination of* $\{b_k(w) : k \in b\text{–supp}(f)\}$.

The following is straightforward.

Lemma 3.3 *For* $f \in \mathrm{Int}(\mathbb{Z})$,

$$\begin{aligned} \min b\text{–supp}(f) &= \min\{k \in \mathbb{N} : f(k) \neq 0\} \\ \max b\text{–supp}(f) &= \deg f \end{aligned}$$

The proof of Theorem 2.2 readily adapts to the following.

Proposition 3.4 *Let* $S \subseteq \mathbb{N}$ *be a finite subset. Let* $\{f_i(w) : i \in S\} \subset \mathrm{Int}(\mathbb{Z})$ *be a set of numerical polynomials such that* $b\text{–supp}(f_i) \subseteq S$ *and* $\deg f_i = i$.

Then the integral matrix $(f_i(j))_{i,j\in S}$ *has determinant* $\prod_{i\in S} f_i^{(i)}(0)$. □

The simplest case to apply this to is when $S = [m, n]$ as an interval in $\mathbb{N}$. Lemma 3.3 gives a simple characterisation as to whether a polynomial $f \in \mathrm{Int}(\mathbb{Z})$ has b–support in $[m, n]$: we need $\deg f \le n$ and $f(k) = 0$ for $0 \le k < m$.

As a special case, given an arbitrary family, say $(f_i(w))$, we can create one with b–supports in $\mathbb{N}_{>0}$ by replacing $f_i(w)$ by $f_i(w) - f_i(0)$. Applying this to the family in Corollary 3.1 leaves the matrix:

$$\begin{bmatrix} q_1 & \binom{n_2+q_2}{2} - \binom{n_2}{2} & \dots & \binom{n_k+q_k}{k} - \binom{n_k}{k} \\ 2q_1 & \binom{n_2+2q_2}{2} - \binom{n_2}{2} & \dots & \binom{n_k+2q_k}{k} - \binom{n_k}{k} \\ \dots & \dots & \dots & \dots \\ kq_1 & \binom{n_2+kq_2}{2} - \binom{n_2}{2} & \dots & \binom{n_k+kq_k}{k} - \binom{n_k}{k} \end{bmatrix}$$

and Proposition 3.4 says that this has determinant 1.

Notice that n_1 is now redundant. We have, essentially, done the first obvious row reduction on the original matrix.

Another interesting sequence to apply this to, although it does not yield a matrix with determinant ± 1, comes from taking products of the $b_i(w)$. Let $f_i(w) = b_{i-m}(w) b_m(w)$ for some fixed m and for $m \le i \le n$.

This satisfies the required conditions with $S = [m, n]$ since $b_m(j) = 0$ for $j < m$. The determinant of the matrix $(f_i(j))_{m \le i, j \le n}$ is then:

$$\prod_{j=m}^{n} \binom{j}{m}.$$

More generally, given a sequence $(k_m, \dots, k_n)$ with $m \le k_i \le i$ define $f_i(w) \coloneqq b_{i-k_m}(w) b_{k_m}(w)$. The matrix $(f_i(j))_{m \le i, j \le n}$ has determinant:

$$\prod_{j=m}^{n} \binom{j}{k_j}.$$

In a similar vein we can consider compositions of the $b_i(w)$. Let $(k_m, \dots, k_n)$ and $(l_m, \dots, l_n)$ be finite sequences of positive integers such that for each j, $k_j l_j = j$ and $l_j \ge m$. Let $f_i(w) = b_{k_i}(b_{l_i}(w))$. This satisfies the required conditions since $b_{l_i}(j) = 0$ for $j < m$ and, if $m > 0$, $b_{k_j}(0) = 0$. The matrix $(f_i(j))_{m \le i, j \le n}$ has determinant:

$$\prod_{j=m}^{n} \frac{j!}{k_j!} \left(\frac{1}{l_j!} \right)^{k_j}.$$

4 The Pascal Transform

An interesting line of further enquiry would be to find natural families of polynomials in $\mathrm{Int}(\mathbb{Z})$ with b–supports of a different form than $[m, n]$, such as a subset of $2\mathbb{N}$. This would be similar to the notion of an *even* function, but that is not as well-behaved a notion in $\mathrm{Int}(\mathbb{Z})$ as it is for general functions due to the fact that we cannot arbitrarily divide by 2. So to get a handle on the b–support of a polynomial we need to find a way to determine the coefficients in its expression as a $\mathbb{Z}$–linear combination of Newton's polynomials.

There are three sets of information about a polynomial in $\mathrm{Int}(\mathbb{Z})$ that completely determine it: the coefficients of the powers, the values at integers, and the coefficients of Newton's polynomials. Of these, the coefficients of the powers have the most complicated structure since they have intricate interdependencies – recall that we're talking about polynomials in $\mathrm{Int}(\mathbb{Z})$ here – so we look at the other two. We'll actually restrict to values at natural numbers. Knowing the coefficients as a combination of Newton's polynomials, we can get the values at natural numbers very simply. Given:

$$f(w) = \sum_{j=0}^{k} c_j b_j(w)$$

then

$$f(n) = \sum_{j=0}^{k} c_j b_j(n) = \sum_{j=0}^{k} c_j \binom{n}{j}$$

This says that the matrix formed by left-justifying Pascal's triangle is the transformation matrix between the coefficients of Newton's polynomials and the sequence of values at natural numbers.

$$\left[\tbinom{i}{j}\right]\left[c_j\right] = \left[f(i)\right]$$

To go in the other direction, we need to invert this matrix. It turns out that the inverse of the matrix $\left[\tbinom{i}{j}\right]$ is very similar to it.

Lemma 4.1 *The inverse of* $\left[\tbinom{i}{j}\right]$ *is* $\left[(-1)^{i+j}\tbinom{i}{j}\right]$.

Proof. We need to show that for any i and j:

$$\sum_k (-1)^{i+k}\binom{i}{k}\binom{k}{j} = \delta_{ij}$$

The terms in the sum are non-zero only when $j \leq k \leq i$. Under these circumstances, we have:

$$\begin{aligned}
\binom{i}{k}\binom{k}{j} &= \frac{i!}{k!(i-k)!}\frac{k!}{j!(k-j)!} \\
&= \frac{i!}{j!}\frac{1}{(i-k)!(k-j)!} \\
&= \frac{i!}{j!(i-j)!}\frac{(i-j)!}{(i-k)!(k-j)!} \\
&= \binom{i}{j}\frac{(i-j)!}{((i-j)-(k-j))!(k-j)!} \\
&= \binom{i}{j}\binom{i-j}{k-j}
\end{aligned}$$

whence summing gives

$$\begin{aligned}
\sum_k (-1)^{i+k}\binom{i}{k}\binom{k}{j} &= (-1)^{i+j}\binom{i}{j}\sum_{j\leq k\leq i}(-1)^{k-j}\binom{i-j}{k-j} \\
&= (-1)^{i+j}\binom{i}{j}\sum_{0\leq r\leq i-j}(-1)^{r}\binom{i-j}{r} \\
&= (-1)^{i+j}\binom{i}{j}(1-1)^{i-j} \\
&= \delta_{ij}
\end{aligned}$$

as required. □

Thus knowing the values $(f(j))$ we get the coefficients c_k via:

$$c_k = \sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}f(j)$$

We can therefore determine the support of a function $f \in \mathrm{Int}(\mathbb{Z})$ by computing the values

$$\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}f(j)$$

with $k \leq \deg f$.

That the inverse of $\left[\binom{i}{j}\right]$ is almost itself is reminiscent of the Fourier transform. Before pursuing that idea, we can adjust it to make it self-inverse. Let (λ_i) be a sequence of integers and let L be the diagonal matrix with the sequence (λ_i) on its main diagonal. Pre-multiplying a matrix by L multiplies the ith row by λ_i. Post-multiplying a matrix by L multiplies the jth column by λ_j. Therefore if $A = \left[a_{ij}\right]$ then $LAL = \left[\lambda_i \lambda_j a_{ij}\right]$. So putting M to be the diagonal matrix with diagonal entries $(-1)^i$ (i.e., alternating 1 and -1), we get that:

$$M\left[\binom{i}{j}\right]M = \left[(-1)^{i+j}\binom{i}{j}\right]$$

and hence:

$$\left[\binom{i}{j}\right]M\left[\binom{i}{j}\right]M = I$$

Therefore, the matrix $\left[(-1)^j\binom{i}{j}\right]$ is self-inverse. We choose to post-multiply by M as the effect of changing the sign of every other column fits better into our scheme: it simply replaces $b_k(w)$ by $(-1)^k b_k(w)$ and therefore in all our analysis simply multiplies the coefficient of $b_k(w)$ by $(-1)^k$.

We can therefore make the following definition:

Definition 4.2 *Define the* Pascal Transform *on* $\mathrm{Map}(\mathbb{N}, \mathbb{Z})$ *by:*

$$P\left[a_j\right] = \left[(-1)^j\binom{i}{j}\right]\left[a_j\right]$$

This then allows us to work with a function via its values or its coefficients of Newton's polynomials as we wish. In particular, if we define the *support* of a sequence $a = \left[a_j\right] \in \mathrm{Map}(\mathbb{N}, \mathbb{Z})$ to be the set $\mathrm{supp}(a) \coloneqq \{j : a_j \neq 0\}$ then we have that the b–support of $\left[a_j\right]$ is the support of $P\left[a_j\right]$ and that:

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathrm{Map}(\mathbb{N}, \mathbb{Z}) : |\mathrm{supp}(P(f))| < \infty\}.$$

5 Vandermonde Matrix

As mentioned in Section 2, the polynomials $(b_k(w))$ form a $\mathbb{Z}$–basis for $\mathrm{Int}(\mathbb{Z})$, a $\mathbb{Q}$–basis for $\mathbb{Q}[w]$, and an $\mathbb{R}$–basis for $\mathbb{R}[w]$. In the latter two cases we have the more standard basis of the monomials. The advantage of the basis $(b_k(w))$ is, as also stated in Section 2, that $b_k(j) = 0$ for $0 \leq j < k$ and $b_k(k) = 1$. This means that the matrix $(b_i(j))$ is triangular with 1s on the main diagonal.

To work with matrices formed by evaluating at other numbers, it is useful to connect the matrix $(b_i(q_j))$ with the Vandermonde Matrix. Up to transposition, this fits Theorem 2.2 with $f_i(w) = w^i$ since:

$$V(q_0, \dots, q_k) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_0 & q_1 & \cdots & q_k \\ \vdots & \vdots & & \vdots \\ q_0^k & q_1^k & \cdots & q_k^k \end{bmatrix}$$

The determinant of this is well-known to be given by:

$$\det V(q_0, \ldots, q_k) = \prod_{0 \le i < j \le k} (q_j - q_i)$$

Theorem 2.2 connects this determinant with that of the matrix formed using the Newton polynomials yielding the formula:

$$\det V(q_0, \ldots, q_k) = \left(\prod_{j=0}^{k} j!\right) \det(b_i(q_j))$$

Putting $q_j = j$, we get

$$\prod_{0 \le i < j \le k} (j - i) = \det V(0, \ldots, k) = \left(\prod_{j=0}^{k} j!\right) \det(b_i(j)) = \prod_{j=0}^{k} j!$$

We therefore get

$$\det(b_i(q_j)) = \prod_{0 \le i < j \le k} \frac{q_j - q_i}{j - i}$$

This nicely complements Theorem 2.2 as it allows us to calculate $\det(b_i(q_j))$ for arbitrary q_j. $\blacksquare$